

WEAK AND STRONG MOMENTS OF RANDOM VECTORS

RAFAL LATAŁA

*Institute of Mathematics, University of Warsaw
Banacha 2, 02-097 Warszawa, Poland
Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-956 Warszawa, Poland
E-mail: rlatala@mimuw.edu.pl*

Abstract. We discuss a conjecture about comparability of weak and strong moments of log-concave random vectors and show the conjectured inequality for unconditional vectors in normed spaces with a bounded cotype constant.

1. Introduction. Let X be a random vector with values in some normed space $(F, \|\cdot\|)$. The question we will discuss is how to estimate $\|X\|_p = (\mathbb{E}\|X\|^p)^{1/p}$ for $p \geq 1$. Obviously $\|X\|_p \geq \|X\|_1 = \mathbb{E}\|X\|$ and for any continuous linear functional φ on F with $\|\varphi\|_* \leq 1$ we have $\|X\|_p \geq (\mathbb{E}|\varphi(X)|^p)^{1/p}$. It turns out that in some situations one may reverse these obvious estimates and show that for an absolute constant C and any $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C \left(\mathbb{E}\|X\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \right).$$

This is for example the case when X has Gaussian or product exponential distribution. In this note we will concentrate on the more general case of log-concave vectors.

A measure μ on \mathbb{R}^n is called logarithmically concave (log-concave in short) if for any compact nonempty sets $A, B \subset \mathbb{R}^n$ and $\lambda \in (0, 1)$,

$$\mu(\lambda A + (1 - \lambda)B) \geq \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

By the result of Borell [3] a measure μ on \mathbb{R}^n with full dimensional support is log-concave if and only if it is absolutely continuous with respect to the Lebesgue measure and has a density of the form e^{-f} , where $f: \mathbb{R}^n \rightarrow (-\infty, \infty]$ is a convex function. Log-concave measures are frequently studied in convex geometry, since by the Brunn-Minkowski inequality

2010 *Mathematics Subject Classification*: Primary 60E15; Secondary 52A40, 60B11.

Key words and phrases: random vectors, log-concave measures, moment estimates.

The paper is in final form and no version of it will be published elsewhere.

uniform distributions on convex bodies as well as their lower dimensional marginals are log-concave. In fact the class of log-concave measures on \mathbb{R}^n is the smallest class of probability measures closed under linear transformation and weak limits that contains uniform distributions on convex bodies. Vectors with logarithmically concave distributions are called log-concave.

In the sequel we discuss the following conjecture posed in a stronger form in [5] about the comparison of strong and weak moment for log-concave vectors.

CONJECTURE 1.1. *For any n dimensional log-concave random vector and any norm $\|\cdot\|$ on \mathbb{R}^n we have for $1 \leq p < \infty$,*

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C_1 \mathbb{E}\|X\| + C_2 \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p}, \quad (1)$$

where C_1 and C_2 are absolute constants.

In Section 2 we gather known results about validity of (1) in special cases. Section 3 is devoted to the unconditional vectors. In particular we show that Conjecture 1.1 is satisfied under additional assumption of unconditionality of X and bounded cotype constant of the underlying normed space.

Notation. Let (ε_i) be a Bernoulli sequence, i.e. a sequence of independent symmetric variables taking values ± 1 . We assume that (ε_i) are independent of other random variables.

By (\mathcal{E}_i) we denote a sequence of independent symmetric exponential random variables with variance 1 (i.e. the density $2^{-1/2} \exp(-\sqrt{2}|x|)$). We set $\mathcal{E} = \mathcal{E}^{(n)} = (\mathcal{E}_1, \dots, \mathcal{E}_n)$ for an n -dimensional random vector with product exponential distribution and identity covariance matrix.

By $\langle \cdot, \cdot \rangle$ we denote the standard scalar product on \mathbb{R}^n and by (e_i) the standard basis of \mathbb{R}^n . We set B_p^n for a unit ball in ℓ_p^n , i.e. $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\}$. For a random variable Y and $p > 0$ we write $\|Y\|_p = (\mathbb{E}|Y|^p)^{1/p}$.

We write C (resp. $C(\alpha)$) to denote universal constants (resp. constants depending only on parameter α). Value of a constant C may differ at each occurrence.

2. Known results. Since any norm on \mathbb{R}^n may be approximated by a supremum of exponential number of functionals we get

PROPOSITION 2.1 (see [5, Proposition 3.20]). *For any n -dimensional random vector X inequality (1) holds for $p \geq n$ with $C_1 = 0$ and $C_2 = 10$.*

It is also easy to reduce Conjecture 1.1 to the case of symmetric vectors.

PROPOSITION 2.2. *Suppose that (1) holds for all symmetric n -dimensional log-concave vectors X . Then it is also satisfied with constants $4C_1 + 1$ and $4C_2$ by all log-concave vectors X .*

Proof. Assume first that X has a log-concave distribution and $\mathbb{E}X = 0$. Let X' be an

independent copy of X , then $X - X'$ is symmetric and log-concave. Moreover for $p \geq 1$,

$$\begin{aligned} (\mathbb{E}\|X\|^p)^{1/p} &= (\mathbb{E}\|X - \mathbb{E}X'\|^p)^{1/p} \leq (\mathbb{E}\|X - X'\|^p)^{1/p}, \\ \mathbb{E}\|X - X'\| &\leq \mathbb{E}\|X\| + \mathbb{E}\|X'\| = 2\mathbb{E}\|X\| \end{aligned}$$

and for any functional φ ,

$$(\mathbb{E}|\varphi(X - X')|^p)^{1/p} \leq (\mathbb{E}|\varphi(X)|^p)^{1/p} + (\mathbb{E}|\varphi(X')|^p)^{1/p} = 2(\mathbb{E}|\varphi(X)|^p)^{1/p}$$

Hence (1) holds for X with constant $2C_1$ and $2C_2$.

If X is arbitrary log-concave then $X - \mathbb{E}X$ is log-concave with mean zero. We have for any $p \geq 1$,

$$(\mathbb{E}\|X\|^p)^{1/p} \leq (\mathbb{E}\|X - \mathbb{E}X\|^p)^{1/p} + \mathbb{E}\|X\|, \quad \mathbb{E}\|X - \mathbb{E}X\| \leq 2\mathbb{E}\|X\|$$

and for any functional φ ,

$$(\mathbb{E}|\varphi(X - \mathbb{E}X)|^p)^{1/p} \leq (\mathbb{E}|\varphi(X)|^p)^{1/p} + |\varphi(\mathbb{E}X)| \leq 2(\mathbb{E}|\varphi(X)|^p)^{1/p}.$$

■

REMARK. Estimating $\|X\|_p$ is strictly connected with bounding tails of $\|X\|$. Indeed by Chebyshev's inequality we have

$$\mathbb{P}(\|X\| \geq e\|X\|_p) \leq e^{-p}$$

and by the Paley-Zygmund inequality and the fact that $\|X\|_{2p} \leq C\|X\|_p$ for $p \geq 1$ we get

$$\mathbb{P}\left(\|X\| \geq \frac{1}{C}\|X\|_p\right) \geq \min\left\{\frac{1}{C}, e^{-p}\right\}.$$

Gaussian concentration inequality easily implies (1) for Gaussian vectors X (see for example Chapter 3 of [8]). For Rademacher sums comparability of weak and strong moments was established by Dilworth and Montgomery-Smith [4]. More general statement was shown in [6].

THEOREM 2.3. *Suppose that $X = \sum_i v_i \xi_i$, where $v_i \in F$ and ξ_i are independent symmetric r.v's with logarithmically concave tails. Then for any $p \geq 1$ inequality (1) holds with absolute constants C_1 and C_2 .*

This immediately implies

COROLLARY 2.4. *Conjecture 1.1 holds under additional assumption that coordinates of X are independent.*

Proof. We have $X = \sum_{i=1}^n e_i X_i$ with X_i independent log-concave real random variables. It is enough to notice that variables X_i have log-concave tails and in the symmetric case apply Theorem 2.3. General independent case may be reduce to the symmetric one as in the proof of Proposition 2.2. ■

The crucial tool in the proof of Theorem 2.3 is the Talagrand two-level concentration inequality for the product exponential distribution [12]:

$$\nu^n(A) \geq \frac{1}{2} \quad \Rightarrow \quad 1 - \nu^n(A + \sqrt{t}B_2^n + tB_1^n) \leq e^{-t/C}, \quad t > 0,$$

where ν is the symmetric exponential distribution, i.e. $d\nu(x) = \frac{1}{2} \exp(-|x|)dx$.

In [5] more general concentration inequalities were investigated. For a probability measure μ on \mathbb{R}^n define

$$\Lambda_\mu(y) = \log \int e^{\langle y, z \rangle} d\mu(z), \quad \Lambda_\mu^*(x) = \sup_y (\langle y, x \rangle - \Lambda_\mu(y))$$

and

$$B_\mu(t) = \{x \in \mathbb{R}^n : \Lambda_\mu(x) \leq t\}.$$

One may show that $B_{\nu^n}(t) \sim \sqrt{t}B_2^n + tB_1^n$. Argument presented in [5, Section 3.3] gives

PROPOSITION 2.5. *Suppose that for some $\alpha \geq 1$ and $\beta > 0$ and any convex symmetric compact set $K \subset \mathbb{R}^n$ we have*

$$\mu(K) \geq \frac{1}{2} \quad \Rightarrow \quad 1 - \mu(\alpha K + B_\mu(t)) \leq e^{-t/\beta}, \quad \text{for all } t > 0. \quad (2)$$

Then inequality (1) holds with $C_1 = \alpha$ and $C_2 = C\beta$.

In [5] it was shown that concentration inequality (2) holds with $\alpha = 1$ for symmetric product log-concave measures and for uniform distributions on B_r^n balls. This gives

COROLLARY 2.6. *Inequality (1) holds with $C_1 = 1$ and universal C_2 for uniform distributions on B_r^n balls $1 \leq r \leq \infty$.*

Modification of Paouris' proof [11] of large deviation inequality for ℓ_2 norm of isotropic log-concave vectors shows that weak and strong moments are comparable in the Euclidean case (see [1] for details):

THEOREM 2.7. *If X is a log-concave n -dimensional random vector then for any Euclidean norm $\|\cdot\|$ on \mathbb{R}^n we have*

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C \left(\mathbb{E}\|X\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \right).$$

3. Unconditional case. We say that a random vector $X = (X_1, \dots, X_n)$ has *unconditional distribution* if the distribution of $(\eta_1 X_1, \dots, \eta_n X_n)$ is the same as X for any choice of signs η_1, \dots, η_n . Random vector X is called *isotropic* if it has identity covariance matrix, i.e. $\text{Cov}(X_i, X_j) = \delta_{i,j}$.

THEOREM 3.1. *Suppose that X is an n -dimensional isotropic, unconditional, log-concave vector. Then for any norm $\|\cdot\|$ on \mathbb{R}^n and $p \geq 1$,*

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C \left(\mathbb{E}\|\mathcal{E}\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \right). \quad (3)$$

Proof. Let $T = \{t \in \mathbb{R}^n : \|t\|_* \leq 1\}$ be the unit ball in the space $(\mathbb{R}^n, \|\cdot\|_*)$ dual to $(\mathbb{R}^n, \|\cdot\|)$. Then $\|x\| = \sup_{t \in T} \langle t, x \rangle$. By the result of Talagrand [13] (see also [14]) there exist subsets $T_n \subset T$ and functions $\pi_n : T \rightarrow T_n$, $n = 0, 1, \dots$ such that $\pi_n(t) \rightarrow t$ for all $t \in T$, $\#T_0 = 1$, $\#T_n \leq 2^{2^n}$ and

$$\sum_{n=0}^{\infty} \|\langle \pi_{n+1}(t) - \pi_n(t), \mathcal{E} \rangle\|_{2^n} \leq C \mathbb{E} \sup_{t \in T} \langle t, \mathcal{E} \rangle = C \mathbb{E} \|\mathcal{E}\|. \quad (4)$$

Let us fix $p \geq 1$ and choose $n_0 \geq 1$ such that $2^{n_0-1} < 2p \leq 2^{n_0}$. We have

$$\|X\| = \sup_{t \in T} \langle t, X \rangle \leq \sup_{t \in T} |\langle \pi_{n_0}(t), X \rangle| + \sup_{t \in T} \sum_{n=n_0}^{\infty} |\langle \pi_{n+1}(t) - \pi_n(t), X \rangle|. \quad (5)$$

We get

$$\begin{aligned} \left(\mathbb{E} \sup_{t \in T} |\langle \pi_{n_0}(t), X \rangle|^p \right)^{1/p} &\leq \left(\mathbb{E} \sum_{s \in T_{n_0}} |\langle s, X \rangle|^p \right)^{1/p} \leq (\#T_{n_0})^{1/p} \sup_{s \in T_{n_0}} (\mathbb{E} |\langle s, X \rangle|^p)^{1/p} \\ &\leq 16 \sup_{t \in T} (\mathbb{E} |\langle t, X \rangle|^p)^{1/p} = 16 \sup_{\|\varphi\|_* \leq 1} (\mathbb{E} |\varphi(X)|^p)^{1/p}. \end{aligned} \quad (6)$$

To estimate the last term in (5) notice that for $u \geq 16$ we have by Chebyshev's inequality

$$\begin{aligned} \mathbb{P} \left(\sup_{t \in T} \sum_{n=n_0}^{\infty} |\langle \pi_{n+1}(t) - \pi_n(t), X \rangle| \geq u \sup_{t \in T} \sum_{n=n_0}^{\infty} \|\langle \pi_{n+1}(t) - \pi_n(t), X \rangle\|_{2^n} \right) \\ \leq \mathbb{P} \left(\exists_{n \geq n_0} \exists_{t \in T} |\langle \pi_{n+1}(t) - \pi_n(t), X \rangle| \geq u \|\langle \pi_{n+1}(t) - \pi_n(t), X \rangle\|_{2^n} \right) \\ \leq \sum_{n=n_0}^{\infty} \sum_{s \in T_{n+1}} \sum_{s' \in T_n} \mathbb{P}(|\langle s - s', X \rangle| \geq u \|\langle s - s', X \rangle\|_{2^n}) \leq \sum_{n=n_0}^{\infty} \#T_{n+1} \#T_n u^{-2^n} \\ \leq \sum_{n=n_0}^{\infty} \left(\frac{8}{u} \right)^{2^n} \leq 2 \left(\frac{8}{u} \right)^{2^{n_0}} \leq 2 \left(\frac{8}{u} \right)^{2p}. \end{aligned}$$

Integrating by parts this gives

$$\begin{aligned} \left(\mathbb{E} \left(\sup_{t \in T} \sum_{n=n_0}^{\infty} |\langle \pi_{n+1}(t) - \pi_n(t), X \rangle| \right)^p \right)^{1/p} \\ \leq \sup_{t \in T} \sum_{n=n_0}^{\infty} \|\langle \pi_{n+1}(t) - \pi_n(t), X \rangle\|_{2^n} \left(16 + \left(2p \int_0^{\infty} u^{p-1} \left(\frac{8}{u+16} \right)^{2p} \right)^{1/p} \right) \\ \leq 32 \sup_{t \in T} \sum_{n=n_0}^{\infty} \|\langle \pi_{n+1}(t) - \pi_n(t), X \rangle\|_{2^n}. \end{aligned} \quad (7)$$

The result of Bobkov and Nazarov [2] gives

$$\|\langle t, X \rangle\|_r \leq C \|\langle t, \mathcal{E} \rangle\|_r \quad \text{for any } t \in \mathbb{R}^n \text{ and } r \geq 1. \quad (8)$$

Thus the statement follows by (4)-(7). ■

REMARK. The only property of the vector X that was used in the above proof was estimate (8). Thus inequality (3) holds for all n -dimensional random vectors satisfying (8).

REMARK. Estimate (8) gives $(\mathbb{E} |\varphi(X)|^p)^{1/p} \leq C (\mathbb{E} |\varphi(\mathcal{E})|^p)^{1/p}$ for any functional φ , therefore Theorem 3.1 is stronger than the estimate from [7]:

$$(\mathbb{E} \|X\|^p)^{1/p} \leq C \mathbb{E} \|\mathcal{E}\|^p \sim C \left(\mathbb{E} \|\mathcal{E}\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E} |\varphi(\mathcal{E})|^p)^{1/p} \right).$$

In some situation one may show that $\mathbb{E}\|\mathcal{E}\| \leq C\mathbb{E}\|X\|$. This is the case of spaces with bounded cotype constant.

COROLLARY 3.2. *Suppose that $2 \leq q < \infty$, $F = (\mathbb{R}^n, \|\cdot\|)$ is a finite dimensional space with a q -cotype constant bounded by $\beta < \infty$. Then for any n -dimensional unconditional, log-concave vector X and $p \geq 1$,*

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C(q, \beta) \left(\mathbb{E}\|X\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \right),$$

where $C(q, \beta)$ is a constant that depends only on q and β .

Proof. Applying diagonal transformation (and appropriately changing the norm) we may assume that X is also isotropic.

By the result of Maurey and Pisier [9] (see also Appendix II in [10]) one has

$$\mathbb{E}\|\mathcal{E}\| = \mathbb{E} \left\| \sum_{i=1}^n e_i \mathcal{E}_i \right\| \leq C_1(q, \beta) \mathbb{E} \left\| \sum_{i=1}^n e_i \varepsilon_i \right\|.$$

By the unconditionality of X and Jensen's inequality we get

$$\mathbb{E}\|X\| = \mathbb{E} \left\| \sum_{i=1}^n e_i \varepsilon_i |X_i| \right\| \geq \mathbb{E} \left\| \sum_{i=1}^n e_i \varepsilon_i \mathbb{E}|X_i| \right\|.$$

We have $\mathbb{E}|X_i| \geq \frac{1}{C}(\mathbb{E}|X_i|^2)^{1/2} = \frac{1}{C}$, therefore

$$\mathbb{E}\|\mathcal{E}\| \leq CC_1(q, \beta) \mathbb{E}\|X\|$$

and the statement follows by Theorem 3.1. ■

For general norm on \mathbb{R}^n one has

$$\mathbb{E}\|\mathcal{E}\| = \mathbb{E} \left\| \sum_{i=1}^n e_i \varepsilon_i |\mathcal{E}_i| \right\| \leq \mathbb{E} \sup_i |\mathcal{E}_i| \mathbb{E} \left\| \sum_{i=1}^n e_i \varepsilon_i \right\| \leq C \log n \mathbb{E} \left\| \sum_{i=1}^n e_i \varepsilon_i \right\|.$$

This together with the similar argument as in the proof of Corollary 3.2 gives the following.

COROLLARY 3.3. *For any n -dimensional unconditional, log-concave vector X , any norm $\|\cdot\|$ on \mathbb{R}^n and $p \geq 1$ one has*

$$(\mathbb{E}\|X\|^p)^{1/p} \leq C \left(\log n \mathbb{E}\|X\| + \sup_{\|\varphi\|_* \leq 1} (\mathbb{E}|\varphi(X)|^p)^{1/p} \right).$$

Acknowledgments. Research of R. Latała was partially supported by the Foundation for Polish Science and MNiSW grant N N201 397437.

References

- [1] R. Adamczak R. Latała, A. Litvak, A. Pajor and N. Tomczak-Jaegermann, in preparation.
- [2] S.G. Bobkov and F.L. Nazarov, *On convex bodies and log-concave probability measures with unconditional basis*, in: Geometric aspects of functional analysis, Lecture Notes in Math. 1807, Springer, Berlin, 2003, 53–69.
- [3] C. Borell, *Convex measures on locally convex spaces*, Ark. Math. 12 (1974), 239–252.

- [4] S.J. Dilworth and S.J. Montgomery-Smith, *The distribution of vector-valued Rademacher series* Ann. Probab. 21 (1993), 2046–2052.
- [5] R. Latała and J.O. Wojtaszczyk, *On the infimum convolution inequality*, Studia Math. 189 (2008), 147–187.
- [6] R. Latała, *Tail and moment estimates for sums of independent random vectors with logarithmically concave tails*, Studia Math. 118 (1996), 301–304.
- [7] R. Latała, *On weak tail domination of random vectors*, Bull. Polish Acad. Sci. Math. 57 (2009), 75–80.
- [8] M. Ledoux and M. Talagrand, *Probability in Banach spaces. Isoperimetry and processes*, Springer-Verlag, Berlin, 1991.
- [9] B. Maurey and G. Pisier, *Séries de variables aléatoires vectorielles indépendantes et propriétés géométriques des espaces de Banach*, Studia Math. 58 (1976), 45–90.
- [10] V.D. Milman and G. Schechtman, *Asymptotic theory of finite-dimensional normed spaces*, Lecture Notes in Math. 1200, Springer-Verlag, Berlin, 1986.
- [11] G. Paouris, *Concentration of mass on convex bodies*, Geom. Funct. Anal. 16 (2006), 1021–1049.
- [12] M. Talagrand, *A new isoperimetric inequality and the concentration of measure phenomenon*, in: Israel Seminar (GAFA), Lecture Notes in Math. 1469, Springer, Berlin, 1991, 94–124.
- [13] M. Talagrand *The supremum of some canonical processes*, Amer. J. Math. 116 (1994), 283–325.
- [14] M. Talagrand, *The generic chaining. Upper and lower bounds of stochastic processes*, Springer, Berlin, 2005.